

Minkowski's Space-Time: From Visual Thinking to the Absolute World*

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Galison traces Minkowski's progression from his visual-geometric thinking to his physics of space-time and finally to his view of the nature of physical reality. Minkowski always held that a sort of "pre-established harmony" existed between mathematics and nature, but then a different sort of "pre-established harmony" than that of Leibniz. Geometry was not merely an abstraction from natural phenomena or a mere description of physical laws through a mathematical construct; rather, the world was indeed a "four-dimensional, non-Euclidean manifold", a true geometrical structure. As a contemporary of Einstein, Minkowski proposed a reconciliation of gravitation and electro-magnetism that he called, "the Theory of the Absolute World". Despite his untimely death, Minkowski holds a prominent place in twentieth century theoretical physics, not the least for his conception of "space-time", emphatically stating that we can no longer speak of "space" and "time", rather "spaces" and "times".

Hermann Minkowski is best known for his invention of the concept of space-time. For the last seventy years, this idea has found application in physics from electromagnetism to black holes. But for Minkowski space-time came to signify much more than a useful tool of physics. It was, he thought, the core of a new view of nature which he dubbed the "Theory of the Absolute World." This essay will focus on two related questions: how did Minkowski arrive at his idea of space-time, and how did he progress from space-time to his new concept of physical reality.¹

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¹ A different approach to Minkowski's work has been taken by Lewis Pyenson in "La Réception de la relativité généralisée: disciplinarité et institutionnalisation en physique": *Revue d'histoire des sciences*, (28 January, 1975):61-73. Pyenson touches on Minkowski's role in the institutionalization of mathematical physics. Pyenson's "Hermann Minkowski and Einstein's Special Theory of Relativity," *Archive for History of Exact Sciences*, 17 (1977):71-95, deals more closely with Minkowski's space-time in special relativity, in order to link Minkowski's work with the institutional history of mathematical physics. Among other sources from the manuscripts at the American Institute of Physics, Pyenson uses the first ten pages of the "Funktionentheorie" lecture discussed in this essay and he provides a valuable transcription of this lecture. The contrast between Minkowski's mathematical physics and the theoretical physics of Einstein and others is further discussed both in Russell McConmach's "Editors Forward" to *Historical Studies in the Physical Sciences*, 7 (Philadelphia: University of Pennsylvania Press, 1976), pp. xi-xxxv, and in Pyenson's article, "Einstein's Early Scientific Collaboration", *Historical Studies in the Physical Sciences*, 7, pp.83-123.

Minkowski was born on 22 June 1864 in Alexotas, Russia. In 1872 his family immigrated to Prussia, where Minkowski attended the Altstädtisches Gymnasium in Königsberg. Like all students of the Gymnasium, he received a strict education which stressed both classical literature and science. Minkowski graduated early and went on to the University of Königsberg, where he studied principally with Wilhelm Weber and Woldemar Voigt. He then spent three semesters in Bonn, returning to Königsberg to receive his doctorate in 1885. As was the custom, he spent several years teaching as a privatdozent in Bonn, where he remained until 1894. After two more years at Königsberg he moved to Zurich for six years. Then, in 1902, at David Hilbert's request, a chair was created for him in Göttingen, where he remained until his untimely death in 1909.²

Minkowski's extraordinary geometric insight was evident by the time he was seventeen, when he won the grand prize of the French Academy for a geometric, general treatment of the theory of quadratic forms.³ Contemporary mathematicians immediately recognized his talent. On receiving his handwritten manuscript of the prize essay, C. Jordan wrote to young Minkowski, "Please work to become an eminent geometrician."⁴ Some years later in 1896, when Minkowski's major work on number theory, *The Geometry of Numbers*, appeared, Hermite wrote to Laugel, "I think I see the promised land."⁵ Harris Hancock put it slightly differently, but no less grandly, in the introduction to his *Development of the Minkowski Geometry of Numbers*: "His grasp of geometrical concepts seemed almost superhuman."⁶ This "almost superhuman" grasp of geometry was put to work in most of Minkowski's mathematical discoveries. One area of application was "geometry of numbers," where he used geometric methods to derive estimates for positive definite ternary forms. Another was to the theory of continued fractions based on the closest packing of spheres. These researches led him to a detailed study of convex bodies which in turn yielded a host of new number-theoretic advances.⁷

Minkowski spoke explicitly about his use of geometrical insight in an unpublished introductory lecture to a number theory course, dated 28 October 1897:

² For biographical details of Minkowski see Minkowski, Hennann, *Hermann Minkowski. Briefe an David Hilbert*, eds. L. Rüdberg and H. Zassenhaus (Berlin: Springer-Verlag, 1973), pp. 9-16.

³ Hancock, Harris, *Development of the Minkowski Geometry of Numbers* (New York: Macmillan, 1973), p. viii.

⁴ Hilbert, David, "Gedächtnisrede, gehalten in der öffentlichen Sitzung der Kgl. Gesellschaft der Wissenschaften zu Göttingen am 1. Mal 1909," *Gesammelte Abhandlungen von Hermann Minkowski* (New York: Chelsea, 1967), p. vii. (author's translation).

⁵ Hancock, *Minkowski Geometry of Numbers*, p. viii (author's translation).

⁶ *ibid.*, p. vii.

⁷ Dieudonné, J., "Hermann Minkowski," *Dictionary of Scientific Biography*, 9 (New York: Scribner, 1975), pp. 411-414 (hereinafter *DSB*).

In [applied number theory] one can frequently make use of geometrical intuition [*geometrischer Anschauung*] for the easier discovery of theorems, and so there arises a field, specific areas of which were first created by Gauss, Dirichlet, Eisenstein, and Hermite, and to which I gave the name Geometry of Numbers. It is therefore essentially a question of using a spatial intuition [*räumlicher Anschauung*] for the uncovering of relations among integers.⁸

Thus geometrical and spatial *Anschauung* form the core of Minkowski's approach to number theory. The distinctive visual cast of the word *Anschauung* (view, outlook, or intuition) is reflected in its scientific usage, both by Minkowski and others, as authors such as Gerald Holton have pointed out.⁹ In this context *geometrische Anschauung* might best be translated as "visual-geometric intuition." Minkowski's claim, then, is that through this visual-geometric intuition we will discover the theorems and relations of number theory. In short, Minkowski's mathematical work was characterized by a self-conscious and successful application of geometrical thinking to fields of mathematics outside of geometry proper.

But Minkowski's interests were not confined to geometry. After receiving his doctorate, he spent five years teaching in Bonn. There he became interested in a variety of problems in physics, and in 1888 published an article on hydrodynamics, which was submitted by Hermann von Helmholtz.¹⁰ He continued in the following years to pursue his interest in physics and reported to Hilbert in 1890 that he was learning both about practical and experimental physics and the work of Helmholtz, J. J. Thomson, and Heinrich Hertz.¹¹ Minkowski's interest in Hertz's work was so strong that he once remarked that had Hertz lived, he (Minkowski) might have turned more completely from mathematics to physics.¹²

⁸ "Zu ihr [die angewandte Zahlentheorie] kann man vielfach von geometrischer Anschauung zur leichteren Auffindung von Sätzen Gebrauch machen und so entsteht ein Gebiet, welches zuerst in einzelnen Partien bei Gauss, Dirichlet, Eisenstein, Hermite auftaucht und welchem ich den Namen Geometrie der Zahlen gegeben habe. Es handelt sich von demselben also wesentlich um einen Gebrauch räumlicher Anschauung zur Aufdeckung von Beziehungen für ganze Zahlen ..."

Minkowski, Hermann, unpublished lecture, 28 October 1897, *MP*, box V, folder 13. See appendix "Notes on Manuscript Sources".

⁹ Holton, Gerald, *Thematic Origins of Scientific Thought: From Kepler to Einstein* (Cambridge, Massachusetts: Harvard University Press, 1973), p. 370, and Arnheim, Rudolf, *Visual Thinking* (Berkeley: University of California Press, 1969), p. 299, discuss *Anschauung* in relation to Pestalozzi's pedagogical use of visual intuition. Holton's analysis of Einstein's visual way of thinking is discussed further in the conclusion of this essay.

¹⁰ "Ueber die Bewegung eines festen Körpers in einer Flüssigkeit," *Sitzungsberichte der Berliner Akademie* 1888, pp. 1095-1110, cited in Hilbert, "Gedächtnisrede," p. xxi.

¹¹ Minkowski, *Briefe an David Hilbert*, pp. 39-40.

¹² Hilbert, "Gedächtnisrede," pp. xxi-xxii.

Minkowski arrived in Göttingen in 1902 during a period of great excitement in the physics community. By 1900, H. A. Lorentz had achieved remarkable success in explaining both the optical and the electrodynamic properties of objects moving in the ether using only electromagnetic considerations.¹³ Wilhelm Wien had responded to these triumphs with a call for a new program for a unified physics. His goal was to bring together mechanics and electrodynamics by explaining mass purely electrodynamically. "It is doubtless one of the most important tasks of theoretical physics," Wien wrote, "to unite the two heretofore completely isolated fields of mechanical and electrodynamic phenomena, and to derive from a common foundation their respective differential equations."¹⁴ Lorentz's 1904 paper caused further excitement by rendering Maxwell's equations covariant to all orders of v/c and thus "explaining" the puzzling Michelson-Morley experiment.¹⁵

At Göttingen these developments were closely watched, for between 1900 and 1910 there were more physicists working on electron theories there than at any other university.¹⁶ Since the time of C. F. Gauss and Wilhelm Weber, Göttingen had been a center for electromagnetic research.¹⁷ In the years following their pioneering work, Eduard Riecke had worked on the electrical properties of metals,¹⁸ and Emil Wiechert,¹⁹ Carl Runge,²⁰ and Arnold Sommerfeld²¹ had conducted further research in electron theory. During Minkowski's career at Göttingen, Max Abraham²² and Walter Kaufmann²³ moved to the forefront of research on the dynamics of the electron. Minkowski himself joined Hilbert in conducting several seminars on the new research in electrodynamics. According to both Hilbert and Max

¹³ On Lorentz's work see McCormach, Russell, "H. A. Lorentz and the Electromagnetic View of Nature," *Isis*, 61 (1970): 459-497; Schaffner, Kenneth, "The Lorentz Theory of Relativity," *American Journal of Physics*, 37 (1969): 498-513; and Miller, Arthur I., "On Lorentz's Methodology," *British Journal for the Philosophy of Science*, 25 (1974): 33ff.

¹⁴ Wien, Wilhelm, "Ueber die Möglichkeit einer elektromagnetischen Begründung der Mechanik," reprinted in *Annalen der Physik*, 5 (1901): 501.

¹⁵ Lorentz, H. A., "Electromagnetic Phenomena in a System Moving with Any Velocity Smaller than that of Light," *Proceedings of the Royal Academy of Amsterdam*, 6 (1904): 809, reprinted in H. A. Lorentz, A. Einstein, H. Minkowski, and H. Weyt, *The Principle of Relativity: A Collection of Original Memoirs on the Special and General Theory of Relativity*, trans. W. Perett and G. B. Jeffrey, nn. by A. Sommerfeld (New York: Dover, 1952), pp. 1-34. See note 25.

¹⁶ Lewis Pynson, "Einstein's Early Collaboration," *Historical Studies in the Physical Sciences*, 7 (1976): 89.

¹⁷ Woodruff, A. E., "Wilhelm Eduard Weber," *DSB*, 14, pp.203-209.

¹⁸ Goldberg, Stanley, "Eduard Riecke," *DSB*, 11, pp.445-447.

¹⁹ Bullen, K. E., "Emil Wiechert," *DSB*, 14, pp.327-328.

²⁰ Forman, Paul, "Carl David Tolme Runge," *DSB*, 11, pp.610-614.

²¹ Forman, Paul and Hermann, Armin, "Arnold Sommerfeld," *DSB*, 12, pp.525-532

²² Goldberg, Stanley, "Max Abraham," *DSB*, 1, pp.23-25. For more on Abraham's work see Goldberg, "The Abraham Theory of the Electron: The Symbiosis of Experiment and Theory," *Archive for History of Exact Sciences*, 7 (1970): 7-25.

²³ Campbell, John T., "Walter Kaufmann," *DSB*, 7, pp.263-265.

Born it was during these seminars that Minkowski first began to develop his new ideas on relativity and space-time.²⁴

The purpose of this paper is to trace Minkowski's progression from his visual-geometric thinking to his physics of space-time, and finally to his view of the nature of physical reality. He held that because of a "pre-established harmony between mathematics and nature", geometry could be used as a key to physical insight. Thus he was able to justify relativity as the physical theory with the more satisfying geometrical structure. But this mathematical structure came to mean more to Minkowski than simply a reformulation of relativity. Minkowski eventually ascribed physical reality to the geometry of space-time.

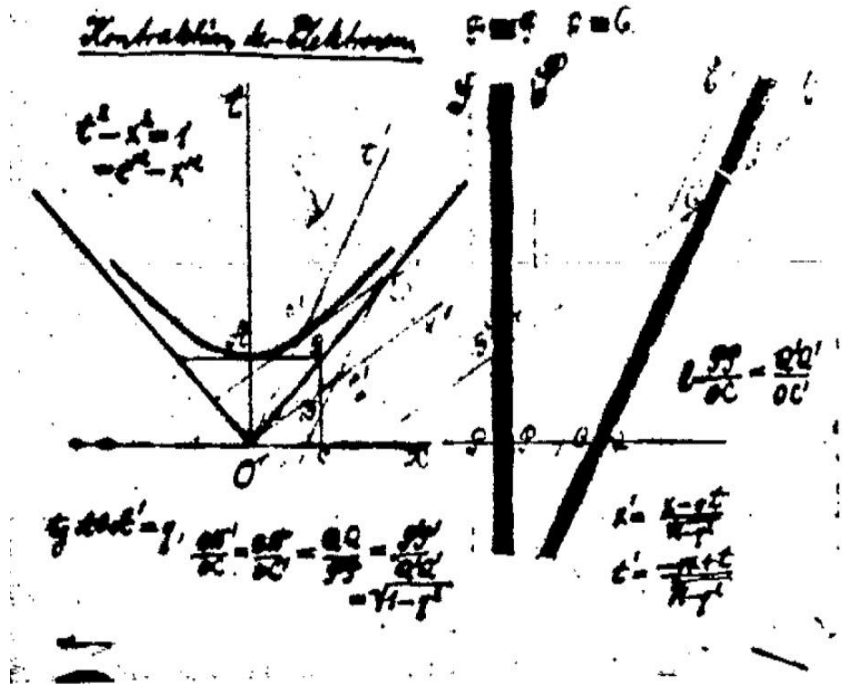


Figure 1: Lecture Slide

Photocopy of slide presented in the lecture "Space and Time", Cologne, 21 September 1908, before the *Versammlung Deutscher Naturforscher und Ärzte*.

For sources on Minkowski's development of space-time we can draw on three public presentations of his views: the speeches "Space and Time" and "The Principle of Relativity", and the paper "The Fundamental Equations of

²⁴ Born, Max, "Erinnerungen an Hermann Minkowski zur 50. Wiederkehr seines Todestages," *Die Naturwissenschaften*, 17 (1959): 502; Hilbert, "Gedächtnisrede," p. xxii.

Electromagnetic Phenomena in Moving Bodies.” “Space and Time” (“*Raum und Zeit*”) was delivered to the eightieth meeting of the Assembly of Natural Scientists and Physicians in Cologne on 21 September 1908. The famous space-time diagram was first seen as a lecture slide (reproduced in Figure 1) and perhaps sketched for the first time in Minkowski’s notes for the talks (Figure 2). The lecture was first printed in 1909 in the *Physikalische Zeitschrift*.²⁵ “The Principle of Relativity” (“*Das Relativitätsprinzip*”) was presented to the Göttingen Mathematische Gesellschaft on 5 November 1907, and published by Arnold Sommerfeld in 1915 in the *Annalen der Physik*, six years after Minkowski’s death.²⁶ The ideas on space and time developed in these lectures were, to some extent, applied in a major work on the laws of electrodynamics, “The Fundamental Equations for Electromagnetic Phenomena in Moving Bodies” (“*Die Grundgleichungen für die elektromagnetische Vorgänge in bewegten Körpern*”), published in 1908.²⁷ The “*Grundgleichungen*” is important for both its results (the first relativistically correct presentation of Maxwell’s equations in a ponderable medium), and its mathematical formalism (tensor calculus). In addition, I have found a variety of unpublished papers relevant to Minkowski’s views on space and time. These are used throughout the paper and are described in the appendix, “Notes on Manuscript Sources.”

Minkowski and the Electromagnetic World Picture

As has been amply discussed in the literature, it was Einstein’s contribution to have abandoned the search for a dynamics of the electron by turning first to the kinematics of macroscopic bodies.²⁸ This involved an epistemological criticism of the concepts of space and time. By provisionally neglecting the goal of finding a dynamics of the structure of matter, he arrived deductively

²⁵ *Physikalische Zeitschrift*, 10 (1909): 104-111, reprinted in H. A. Lorentz *et al.*, *Das Relativitätsprinzip*, 5th ed. (Stuttgart: B. G. Teubner, 1974), pp. 54-56, hereinafter RZ. This edition has been translated as *The Principle of Relativity* (see note 15), hereinafter ST (for “Standard Translation”); the article “Space and Time” in this edition is the standard translation of “*Raum und Zeit*.” References will be first to the German and then to the English, e.g., RZ, p. 54; ST, p. 75. English translations are from ST except where 947 sciences-10: 58.

²⁶ *Annalen der Physik*, 47 (1915): 927-938 (hereinafter RP). Translations from RP are the author’s.

²⁷ *Nachrichten der Königlichen Gesellschaft der Wissenschaft und der Georg-August Universität zu Göttingen, Mathematisch-physikalische Klasse* (1908), pp. 53-111, reprinted in Hilbert, *Abhandlungen von Minkowski*, pp. 352-404.

²⁸ On the Electromagnetic World Picture see Holton, *Thematic Origins*, pp. 177-179; Schaffner, K. F., “Lorentz Theory,” *op. cit.* (note 13), esp. pp. 508-513; McConnmach, Russell, “Einstein, Lorentz and the Electron Theory,” *Historical Studies in the Physical Sciences*, 2 (1970): 69-81; Miller, Arthur I., “A Study of Henri Poincaré’s ‘Sur la Dynamique de l’électron,’” *Archive for History of Exact Sciences*, 10 (1973): 207-328, esp. 207-233; Miller, Arthur I., “The Physics of Einstein’s Relativity Paper of 1905 and the Electromagnetic World Picture of 1905” *American Journal of Physics*; Hirose, Tetu, “Theory of Relativity and the Ether,” *Japanese Studies in the History of Science*, 7 (1968): 37-58; and the items in note 13.

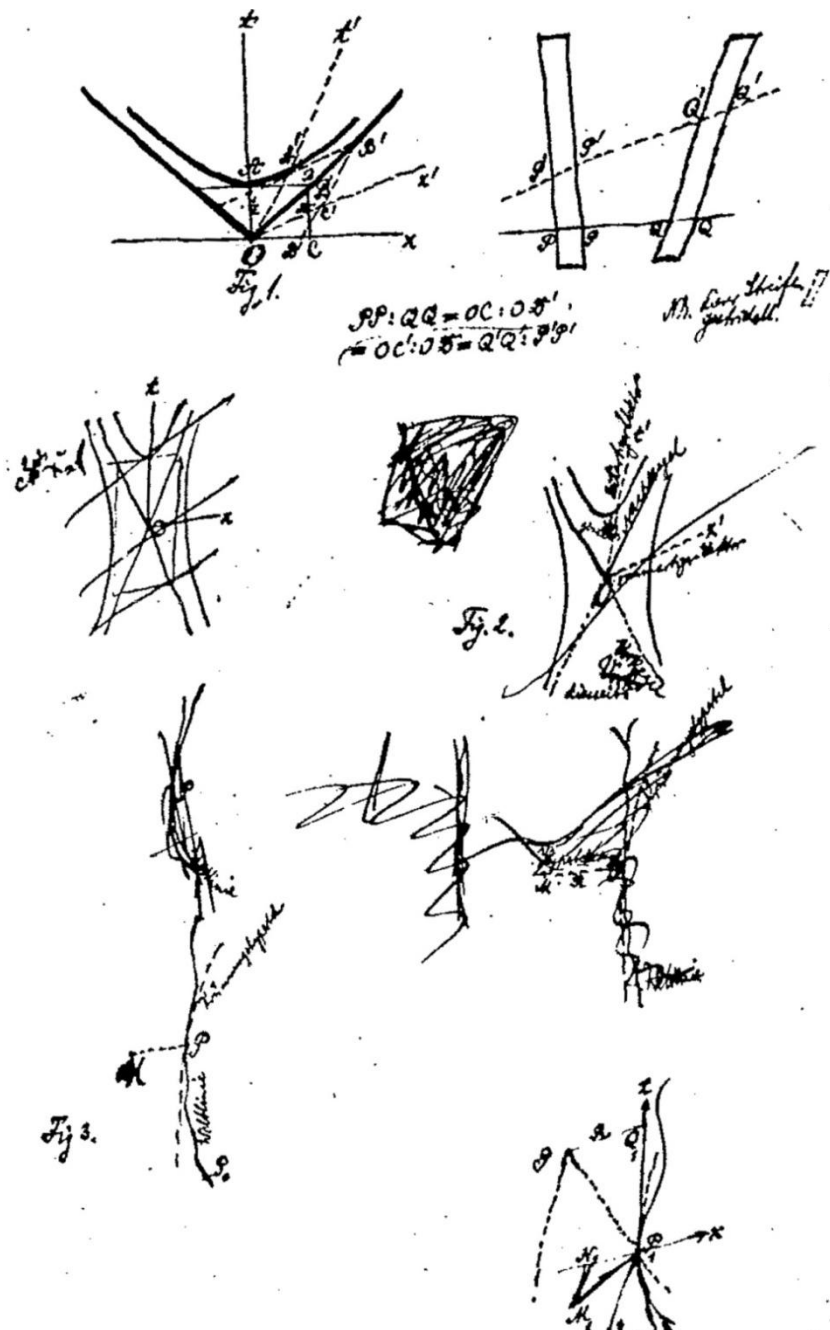


Figure 2: Early Space-Time Diagrams.

These may be the first space-time diagrams ever drawn. From the Göttingen Archives Draft RZ 2, p. 10, unlabeled.

at the space and time transformations which Lorentz had adopted without justification. Conversely, Einstein's explanation of the null result of the optical experiments followed directly from his axioms, while Lorentz and Henri Poincaré had to explain these results with hypotheses on matter and forces. The Significance of Einstein's work was not, however, immediately clear to physicists of the day. While Einstein's theory of special relativity was published in 1905, it had just begun to gain acceptance in 1910.²⁹ Instead, physicists like Poincaré, Lorentz, and Abraham continued to work towards the goal of explaining all of mechanics purely electro-dynamically in accordance with the reductionist view which held the electron to be the fundamental building block of all matter. In fact, all three of these scientists continued to search for an electromagnetic explanation of mass until their respective deaths.

Minkowski's view on the question of reductionism is apparent in "The Principle of Relativity," where he writes, "Here we find ourselves at a standpoint where the true physical laws are not yet completely known to us. One day, perhaps, a reduction will be possible based on purely electrical consideration....,"³⁰ or again in the draft of the introductory lecture on complex analysis, where he remarks, "Physicists suspect that one day it might be possible to explain all natural phenomena purely electro-dynamically so that finally there will be no matter, nothing but electricity in the world."³¹ Assertions such as these make it clear that Minkowski did not break with tradition in order to follow Einstein's new approach. In fact, when Minkowski refers to Einstein, it is evident that he saw Einstein as furthering the Electromagnetic World Picture. He asserts, for example, that "from this very strange sounding hypothesis [the Lorentz contraction hypothesis], the Postulate of Relativity was finally developed in a form which is exceptionally accessible to the mathematician. We owe the working out of the general principle to Einstein, Poincaré and Planck, whose work I will shortly consider more closely."³² This evaluation of Einstein's role is voiced again in an unpublished draft of "The Relativity Principle": "As to the merits of the individual authors: the (essential) foundation of the ideas originates with Lorentz; Einstein more cleanly developed the Principle of Relativity. At the

²⁹ Holton, *Thematic Origins*, pp. 268-269.

³⁰ "Hier stellen wir uns auf den Standpunkt, die zutreffenden physikalischen Gesetze sind uns noch nicht völlig bekannt. Eines Tages würde vielleicht eine Zurückführung auf reine Elektrizitätslehre möglich sein..." *RP, op. cit.*, p. 931.

³¹ "Nun schwebt den Physikern der Gedanke vor, dass es eines Tages gelingen möchte, alle Naturvorgänge rein elektrodynamisch zu erklären, so dass es schliesslich auf der Welt nichts anderes als Elektrizität, keine Materiel gieht." *Lecture on complex analysis*, pp. 11-12. See appendix "Notes on Manuscript Sources" for details.

³² " Aus dieser höchst seltsam klingenden Hypothese hat sich dann schliesslich das Postulat der Relativität in einer Form herausentwickelt, die dem Verständnis des Mathematikers besonders gut zugänglich ist. Verdienste um die Ausarbeitung des allgemeinen Prinzips haben Einstein, Poincaré und Planck, über deren Arbeiten ich alsbald Näheres sagen werde." *RP, op. cit.*, p. 928.

same time he applied it with special success to the treatment of special problems of the optics of moving media, and finally he was also the first to draw the consequences of the variability of mechanical mass in thermodynamic processes.”³³ Minkowski, like Lorentz, took the transformations to be an explanation of phenomena, whereas Einstein considered them as a consequence of our measurement of space and time. It is thus understandable that Einstein's work appeared to Minkowski as a generalization of Lorentz's. Minkowski made no mention, to this point, of Einstein's critical contribution to the understanding of the physical significance of the transformations.

With this background, one is not surprised at Minkowski's assessment of his own contribution to relativity theory, which he refers to as the “World Postulate” (to be discussed below). This judgment occurs at the end of “Space and Time,” where Minkowski returns to mechanics to present it in “harmony” with electrodynamics: “The validity without exception of the world-postulate, I like to think, is the true nucleus of an electromagnetic image of the world, which, discovered by Lorentz, and further revealed by Einstein, now lies open in the full light of day.”³⁴ In sum, Minkowski still hoped for the completion of the Electromagnetic World Picture through relativity theory. Moreover, he saw his own work as completing the program of Lorentz, Einstein, Planck, and Poincaré. Of these, it was Poincaré who most directly influenced the mathematics of Minkowski's space-time.

As Minkowski acknowledges many times in “The Principle of Relativity”, his concept of space-time owes a great deal to Poincaré's work.³⁵ “*Sur la Dynamique de l'électron*” contains Poincaré's systematic search for the invariants of the Lorentz transformation. But of even greater significance

³³ “Was das Verdienst der einzelnen Autoren angeht, so rühren die {wesentlichen} Grundlagen der Ideen von Lorentz her, Einstein hat das Prinzip der Relativität reinlicher herauspräpariert, zugleich es mit besonderem Erfolge zur Behandlung spezieller Probleme der Optik bewegter Medien angewandt, endlich auch zuerst Folgerungen über Veränderlichkeit der mechanischen Masse bei thermodynamischen Vorgängen gezogen.” Draft RP A, p. 16. See appendix “Notes on Manuscript Sources” for details. Crossed brackets indicate that the enclosed work was crossed out in the original. At this early time, (1907) it is clear that Minkowski did not understand the import of Einstein's theory. It is therefore surprising to read in Max Born's *Autobiography* (New York: Charles Scribner's Sons, 1978), p. 131, that “[Minkowski] told me later that it came to him as a great shock when Einstein published his paper in which the equivalence of the different local times of observers was pronounced; for he had reached the same conclusions independently but did not wish to publish them because he wished first to work out the mathematical structure in all its splendour.” My thanks to Professor I. B. Cohen for showing me this quotation.

³⁴ “Die ausnahmslose Gültigkeit des Weltpostulates ist, so möchte ich glauben, der wahre Kern eines elektromagnetischen Weltbildes, der von Lorentz getroffen, von Einstein weiter herausgeschält, nachgerade vollends am Tage liegt.” *op. cit.*, RZ, p. 66; ST, p. 91.

³⁵ Poincaré, Henri A., “*Sur la Dynamique de l'électron*”, *Rendiconti del Circolo Matematico di Palermo*, 21 (1906): 129-175; partial translation in W. Kilmister, ed.; *Special Theory of Relativity* (Oxford: Pergamon, 1970), pp. 144-185. For a critical work on this paper see Miller, “A Study of Poincaré's ‘*Sur la Dynamique*’,” *op. cit.*

to Minkowski was Poincaré's four-dimensional x, y, z, ict -space, which contains the seeds of the four vector formalism. If, Poincaré writes,

$$\begin{array}{cccc} x & y & z & t\sqrt{-1} \\ \delta x & \delta y & \delta z & \delta t\sqrt{-1} \\ \delta_1 x & \delta_1 y & \delta_1 z & \delta_1 t\sqrt{-1} \end{array}$$

are regarded as the coordinates of three points P, P', P'' in four-dimensional space, we see that the Lorentz transformation is simply a rotation of this space about a fixed origin. The only distinct invariants are therefore the six distances of the points P, P', P'' from one another and from the origin, or alternatively the two expressions

$$x^2 + y^2 + z^2 - t^2, \quad x\delta x + y\delta y + z\delta z - t\delta t$$

and the four expressions of the same form obtained by permuting the three points P, P', P'' in any manner.³⁶

Minkowski clearly draws on the interpretation both of the invariants as distances and of the Lorentz transformations as rotations in the x, y, z, ict -space. Notice that by giving the fourth coordinate the dimensions of ict , Poincaré, in contrast to Minkowski, does not emphasize the non-Euclidean nature of the space.

From these invariants Poincaré was able to construct a covariant law of gravitation consistent with special relativity. But Poincaré ascribed neither metaphysical nor physical importance to the four dimensional representation. Indeed, as late as 1908 he asserted:

It seems in fact that it would be possible to translate our physics into the language of geometry of four dimensions; to attempt this translation would be to take great pains for little profit, and I shall confine myself to citing the mechanics of Hertz where we have something analogous. However, it seems that the translation would always be less simple than the text, and that it would always have the air of a translation, that the language of three dimensions seems the better fitted to our description of the world although this description can be rigorously made in another idiom.³⁷

³⁶ Poincaré, "Sur la Dynamique," in Kilmister, *op. cit.*, pp. 175-76.

³⁷ Poincaré, Henri, *Science and Method*, trans. John W. Boulduc (New York: Dover, 1959), p. 427.

By contrast Minkowski, beginning with precisely the same formalism, came to believe that the laws of physics would only be fully understood in four-dimensional space-time.

The Union of Space and Time

To understand Minkowski's idea of space-time, it is revealing to compare and contrast the introductions to his two lectures, "The Principle of Relativity" and "Space and Time." "The Principle of Relativity" begins with a discussion of the new notions of space and time which follow from relativity theory: "Out of the electromagnetic theory of light there recently seems to have come a complete transformation of our representations [*Vorstellungen*] of space and time which must be of exceptional interest for the mathematician to learn."³⁸ Minkowski's use of the term *Vorstellung* announces the orientation of his discussion. *Vorstellung* is an abstract term of concrete origin. Literally a "placing before," it has a more substantial connotation than its English translation "representation," "idea," or "conception." Aware of Minkowski's background, we can see his "spatial intuition" used from the outset. More explicitly, Minkowski next stresses the mathematical interest of the new theory:

The mathematician is also especially well prepared to pick up the new views [*Anschauungen*] of space and time because it involves acclimating himself to conceptual schemes [*Begriffsbildungen*] with which he has long been familiar. The physicist, on the other hand, must discover afresh these concepts [*Begriffe*], and must painfully cut his way through a jungle of obscurities. Meanwhile close by, the old, excellently laid out path of the mathematician comfortably leads forward.³⁹

Where the mathematician uses views (*Anschauungen*) and conceptual schemes (*Begriffsbildungen*), the physicist must struggle with concepts (*Begriffe*). The distinct contrast between the visual terms employed to describe mathematics and the more formal terminology used for physics indicates that Minkowski attaches particular significance to his geometrical approach to relativity: "Above all, the new formulation would be, if in fact it

³⁸ "Von der elektromagnetischen Lichttheorie ausgehend, scheint sich in der jüngsten Zeit eine vollkommene Wandlung unserer Vorstellungen von Raum und Zeit vollziehen zu wollen, die kennen zu lernen für den Mathematiker jedenfalls von ganz besonderem Interesse sein muss." RP, *op. cit.*, p. 927.

³⁹ "Auch ist er besonders gut prädisponiert, die neuen Anschauungen aufzunehmen, weil es sich dabei um eine Akklimatisierung an Begriffsbildungen handelt, die dem Mathematiker längst äusserst geläufig sind, während die Physiker jetzt diese Begriffe zum Teil neu erfinden und sich durch einen Urwald von Unklarheiten mühevoll einen Pfad durchholzen müssen, indessen ganz in der Nähe die längst vortrefflich angelegte Strasse der Mathematiker bequem vonwärts führt." RP, *op. cit.*, p. 927.

correctly reflects the phenomena, practically the greatest triumph applied mathematics has ever shown. Expressed as briefly as possible, it is this - the world in space and time in a certain sense is a four-dimensional, non-Euclidean manifold"⁴⁰ Thus Minkowski introduces the four-dimensional space before discussing Einstein's requirement that electrodynamic theory yield a symmetry with respect to transformation of inertial reference systems. Instead of emphasizing the physical basis of relativity, Minkowski stresses the underlying mathematics of geometry. For Minkowski it is not that physical laws can be equivalently expressed through a mathematical construct, but rather that "in a certain sense...the world *is* a four-dimensional, non-Euclidean manifold" (emphasis added).

This being the case, it is the mathematician who is in a position to say something fundamental about reality, rather than the physicist. In short, Minkowski assigns two different but complementary roles to mathematics. First, mathematics offers physics a set of geometric concepts useful in approaching the new relativity theory. Second, Minkowski identifies physical reality, at least "in a certain sense" with mathematical structure. Should the mathematical statements about reality be valid, "it would be revealed, to the fame of the mathematician and to the boundless astonishment of the rest of mankind, that mathematicians, purely in their imagination, have created a large field to which one day the fullest real existence should be ascribed (though it was never the intention of these idealistic fellows)."⁴¹ Here again, he takes mathematics to be reality rather than an abstraction from or an idealization of reality. If the world is a mathematical structure, then the physical-geometrical laws which describe it acquire an ontological status.

The first paragraph of "The Principle of Relativity," which begins with the first citation in this section and ends with the one above, introduces Minkowski's visual-geometric way of thinking. It stresses the importance he places on the mathematics of relativity and on his view that physical reality lies in four dimensions. These convictions are pointedly reemphasized in Minkowski's introduction to his second and most famous lecture, entitled "Space and Time." That lecture begins, however, with an unexpected tribute to the importance of experiment. "Gentlemen! The views of space and time which I wish to lay before you," he writes, "have sprung from the soil of

⁴⁰ "Überhaupt würden die neuen Ansätze, falls sie tatsächlich die Erscheinungen richtig wiedergehen, fast den grössten Triumph bedeuten, den je die Anwendung der Mathematik gezeitigt hat. Es handelt sich, so kurz wie möglich ausgedrückt - Genauerer werde ich alsbald ausführen - darum, dass die Welt in Raum und Zeit in gewissem Sinne eine vierdimensionale nichteuklidische Mannigfaltigkeit ist." RP, *op. cit.*, p. 927.

⁴¹ "Es würde zum Ruhme der Mathematiker, zum grenzenlosen Erstaunen der übrigen Menschheit offenbar werden, dass die Mathematiker rein in ihrer Phantasie ein grosses Gebiet geschaffen haben, dem, ohne dass dieses je in der Absicht dieser so idealen Gesellen gelegen hätte, eines Tages die vollendetste reale Existenz zukommen sollte." RP, *op. cit.*, pp. 927-928.

experimental physics, and therein lies their strength. They are radical."⁴² Here Minkowski seems to feel obliged to defend his departure from the physics of his time which dealt with specific experimental data. The prominent place Minkowski gives experimental physics is surprising, since it is the only reference to experiment in the speech; he may have felt obligated to acknowledge the role of experiment at least once to avoid the charge of being overly speculative.

The next sentence contains, without doubt, Minkowski's most memorable remark: "Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve independence."⁴³ This sentence in its draft version reads: "Then, from now on, space for itself and time for itself should sink completely into shadows. Only a concept [*Begriff*] obtained by a fusion of the two [concepts] will show a free existence. I will provisionally call this the absolute (?) world until a clear and distinct label for this notion can be found."⁴⁴ The published version refers to the "independence" of space-time, the draft form to its "free existence." Both paragraphs emphasize that Minkowski's new space-time exists independently of observer, unlike the old concepts of space and time. It is this freedom from reference frame which entitles the four-dimensional space to be called "The Absolute World."⁴⁵

The draft manuscripts of the "Space and Time" lecture reveal that composing this crucial first page was a struggle for Minkowski; words are crossed out two or three times, phrases are eliminated, replaced, and struck out again (see Figure 3). The terms used in this introduction were very carefully chosen and indicate the importance Minkowski attached to aspects other than the formal results of his investigations. Consider the published sentence, "*Ihre Tendenz ist eine radikale.*" The choice of words would seem

⁴² "M. H.! Die Anschauungen über Raum und Zeit, die ich Ihnen entwickeln möchte, sind auf experimentell-physikalischem Boden erwachsen. Darin liegt ihre Stärke. Ihre Tendenz ist eine radikale." *op. cit.*; RZ, p. 54; ST, p. 75.

⁴³ "Von Stund an sollen Raum für sich und Zeit für sich völlig zu Schatten herabsinken und nur noch eine Art Union der beiden soll Selbständigkeit bewahren." *op. cit.*, RZ, p. 54; ST, p. 75.

⁴⁴ "Dann, von Stund an, sollen Raum für sich und Zeit für sich zu völligen Schatten herabsinken, nur noch ein durch Verschmelzung der Beiden gewonnener Begriff, den ich provisorisch bis zur Ersinnung eines eigenartigeren den Namen absolute (?) Welt geben will, wird eine freie Existenz zeigen." Draft RZ 2, p. 1, labeled I (Figure 3).

⁴⁵ The standard translation may have led to some misunderstanding on this point because it renders the ending of the above citation as, "...only a kind of union of the two will preserve an 'independent reality'" (emphasis added); *Selbständigkeit* should be simply "independence." Hans Reichenbach has commented on this confusion: "The first part of Minkowski's remark has unfortunately caused the erroneous impression that all visualizations of time as time and space as space must disappear." Reichenbach, Hans, *The Philosophy of Space and Time* (New York: Dover, 1957), p. 160.

more appropriate to a political tract than a discussion of physical theory, yet in the draft the sentence is even stronger. The character of his new views on space and time, Minkowski writes, "is mightily revolutionary, to such an extent that when they are completely accepted, as I expect they will be, it will be disdained to still speak about the ways in which we have tried to understand space and time."⁴⁶ The theme of the opening paragraph to "Space and Time" is thus closely related to that of "The Principle of Relativity." Where "Space and Time" speaks of "a kind of union" of space and time, "The Principle of Relativity" presents the world as a "four-dimensional manifold." Although the wording is different to accommodate the different audiences of the two lectures, the idea is the same: beyond the divisions of time and space which are imposed on our experience, there lies a higher reality, changeless, and independent of observer.

The Pre-Established Harmony between Mathematics and Physics

Minkowski hoped to reformulate the physics of Einstein, Lorentz, and Poincaré to yield a "world" with a "free existence." But clearly it would not be enough for Minkowski to use the fact that he thought geometrically to justify a transition of our concepts of space and time. Instead, he grounds his belief in the truth-revealing power of mathematics (geometry in particular) in what he calls the "pre-established harmony between mathematics and physics." This allows him to isolate and investigate mathematical elements of a theory with the faith that in coming back to physical reality the results will be valid and fruitful.

Unfortunately, Minkowski does not discuss the philosophical origins and implications of the "pre-established harmony"; however, his references to it abound. For instance, in the lectures on complex analysis Minkowski asserts that "there emerges a pre-established harmony among the current mathematical branches of knowledge, that is, the specific conceptual schemes [*Begriffsbildungen*] and problems which have proven themselves valuable for the further development of the theory also prove themselves to be fundamental through the development of physical theories."⁴⁷ And in the conclusion to "Space and Time", Minkowski assures the reader that "in the development of its mathematical consequences there will be ample suggestions for experimental verifications of the...[relativity principle], which will suffice to conciliate even those to whom the abandonment of old

⁴⁶ "Ihr Character ist höchst gewaltig revolutionär, derart, dass wenn sie durchdringen, woran ich glaube, es verpönt sein wird, noch davon zu sprechen wie wir bislang uns Mühe gaben, Raum und Zeit zu verstehen." Draft RZ 2, p. 1, labeled 1(Figure 3).

⁴⁷ "Es tritt darin eine prästabilierte Harmonie der mathematischen Wissenszweige zu Tage, dass die nämlichen Begriffsbildungen und Probleme, welche für die Weiterführung der Theorie sich als wertvoll erweisen, auch durch die Entwicklung der physikalischen Theorien sich als fundamental aufdrängen," *Introductory lecture on complex analysis, op cit*, p. 7.

established views is unpleasant or painful, by the idea of a pre-established harmony between pure mathematics and physics.”⁴⁸

References to the pre-established harmony appear throughout the draft versions of “Space and Time” as well. In one draft, Minkowski refers to the “pre-established harmony between pure mathematics and nature.”⁴⁹ But perhaps the most striking discussion of his belief in the power of mathematics to lead us to an understanding of physical reality appears in another draft of “Space and Time”: “Electrical theory seems, like no second branch of physics, to be predisposed...for triumphs of pure mathematics. In the world of pure ether, the most fragile mathematical structures $\|$ -*Bildungen* $\|$ seem to attain complete $\|$ -reality $\|$ life, whereas everywhere else (one thinks for example of hydrodynamics) the mathematical formulation and problems prove to be only a distant, idealized approximation to crude reality.”⁵⁰ For Minkowski the investigation of pure mathematics was fundamentally tied to a search for truth in the physical world, and it is mathematical *Bildungen* - literally pictures - which will reveal nature’s secrets.

Minkowski’s faith in the pre-established harmony dictated the structure and emphasis of his relativity work. The text of “Space and Time,” for example, applies the pre-established harmony immediately after the introduction. “First of all, I should like to show how it might be possible,” Minkowski began, “setting out from the accepted mechanics of the present day, along a purely mathematical line of thought, to arrive at changed ideas of space and time.”⁵¹ Physicists ignore the geometry implicit in physics, he claims, perhaps because by the time they come to study mechanics, they no longer question the axioms of elementary geometry.⁵² Minkowski, however, despite his earlier remark that the new physics rests on the ground of “experimental physics,” immediately reveals that his interests are obviously mathematical. In any case, “the equations of Newton’s mechanics exhibit a two-fold invariance. Their forms remain unaltered, firstly, if we subject the

⁴⁸ “Bei der Fortbildung der mathematischen Konsequenzen werden genug Hinweise auf experimentelle Verifikationen des Postulates sich einfinden, um auch diejenigen, denen ein Aufgeben altgewohnter Anschauungen unsympathisch oder schmerzlich ist, durch den Gedanken an eine prästabilierte harmonie zwischen der reinen Mathematik und der Physik auszusöhnen.” *op. cit.*, RZ, p. 66; ST, p. 91.

⁴⁹ Draft RZ 4, p. 22, labeled 20.

⁵⁰ “Die Elektrizitätslehre scheint wie kein zweites Gebiet der Physik prädisponiert...für Triumphe der reinen Mathematik. Während anderwärts, man denke z. B. an die Hydrodynamik, die mathematischen Formulierungen und Problemstellungen nur als entfernte ideale Annäherungen an die rohe Wirklichkeit sich erweisen, scheinen in der Welt des reinen Äthers die zartesten mathematischen Bildungen vollkommenes $\|$ -Realität $\|$ Leben zur erlangen.” Draft RZ 4, p. 1, labeled 1. $\|$ -Crossed brackets $\|$ indicate the enclosed word was crossed out in the original.

⁵¹ “Ich möchte zunächst ausführen, wie man von der gegenwärtig angenommenen Mechanik wohl durch eine rein mathematische Überlegung zu veränderten Ideen über Raum und Zeit kommen könnte.” *op. cit.*, RZ, p. 54; ST, p.75.

⁵² *ibid.*

underlying system of spatial coordinates to any arbitrary *change of position*; secondly, if we change its state of motion, namely, by imparting to it any *uniform translatory motion*; furthermore, the zero point of time is given no part to play.”⁵³ Minkowski then explains that the first group of transformations shows the invariance of the form of the equations of motion under displacements or rotations of the coordinate system. This group's validity is based on two geometric presuppositions about space: homogeneity and isotropy. The invariance of the second group, that of uniform translation, is a physical property based on the assumption that there is no mechanical phenomenon which allows us to distinguish a preferred inertial system. On the one hand, we have a group of geometric transformations; on the other, a group of physical ones: “Thus the two groups, side by side, lead their lives entirely apart. Their utterly heterogeneous character may have discouraged any attempt to compound them. But it is precisely when they are compounded that the complete group, as a whole, gives us to think.”⁵⁴

The goal of Minkowski's investigation is to understand the “complete group.” To this end, “we want to visualize the relationships graphically.” If we let x, y, z be the space coordinates and t be time, we can represent a point of space at a particular time by (x, y, z, t) . The object given by a particular value of such a quadruple Minkowski calls a “world-point.” The collection of all world-points constitutes the world, and the world-points that trace a single object's existence in space and time are christened its “world-line.”

Though an appreciation of four dimensions seems difficult to the uninitiated, Minkowski insists that “with this most valiant piece of chalk I might project upon the blackboard four world-axes. Since merely one chalky axis, as it is, consists of molecules all a-thrill, and moreover is taking part in the earth's travels in the universe, it already affords us ample scope for abstraction; the somewhat greater abstraction associated with the number four is for the mathematician no infliction.”⁵⁵ The mathematician's four dimensions are no less real for Minkowski than the physicist's three. In fact, the invariance and timelessness of the world-line give Minkowski reason to believe that the four-dimensional manifold is a higher reality than the three

⁵³ *ibid.* “Die Gleichungen der Newtonschen Mechanik zeigen eine zweifache Invarianz. Einmal bleibt ihre Form erhalten, wenn man das zugrunde gelegte räumliche Koordinatensystem einer beliebigen *Lagenveränderung* unterwirft, zweitens, wenn man es in seinem Bewegungszustande verändert, nämlich ihm irgendeine *gleichförmige Translation* aufprägt; auch spielt der Nullpunkt der Zeit keine Rolle.”

⁵⁴ “So führen jene zwei Gruppen ein völlig getrenntes Dasein nebeneinander. Ihr gänzlich heterogener Charakter mag davon abgeschreckt haben, sie zu komponieren. Aber gerade die komponierte volle Gruppe als Ganzes gibt uns zu denken auf.” *op. cit.*, RZ, p. 54; ST, p. 76.

⁵⁵ Ich könnte mit kühner Kreide vier Weltachsen auf die Tafel werfen. Schon *eine* gezeichnete Achse besteht aus lauter schwingenden Molekülen und macht zudem die Reise der Erde im All mit, gibt also bereits genug zu abstrahieren auf; die mit der Anzahl 4 verbundene etwas grössere Abstraktion tut dem Mathematiker nicht wehe.” *op. cit.*, RZ, p. 55; ST, p. 76.

dimensions we perceive. Then, almost as if to reassure us of the physical reality of the manifold, Minkowski immediately adds, "Not to leave a yawning void anywhere, we will imagine that everywhere and at every time there is something perceptible. To avoid saying 'matter' or 'electricity' I will use for this something the word 'substance'."⁵⁶ When Minkowski places a "substance" at every point in space-time, he is looking back to the ponderomotive ether, refusing to discard the concept entirely although it has been voided of qualities and properties.

Out of these considerations came Minkowski's most prophetic comment, which implies that physical law will someday be expressible as laws of world-lines. "The whole universe is seen to resolve itself into similar world-lines, and I would fain anticipate myself by saying that in my opinion physical laws might find their most perfect expression as reciprocal relations between these world-lines."⁵⁷ Although Minkowski's prediction did not anticipate the field-theoretical direction general relativity would eventually take, his view has had far-reaching consequences, especially for the theory of interacting particles.

The Role of Aesthetic Criteria

If one grants that Minkowski can pass from good mathematics to productive physics, it remained for him to ground the new physics on mathematics alone. He accomplishes this by comparing Newtonian and relativistic theories on the basis of three criteria of geometrical elegance that emerge from his visual thinking: symmetry, generality, and invariance. Together they seem to form the motivation and the justification for Minkowski's adoption of the new physics.

Symmetry

In "The Principle of Relativity," immediately after the introduction and brief remarks on the Michelson-Morley experiment, Minkowski turns to the question of symmetry. Symmetry plays a complex and vital role in the development of relativity; it is therefore important to distinguish between a variety of concepts which fall under its name. The symmetry Minkowski wishes to point out emerges from the basic equations of the Lorentz theory of the electron, which "possess even a further symmetry (other than that they are independent of any particular orthogonal coordinate system in space). In the

⁵⁶ "Um nirgends eine gähnende Leere zu lassen, wollen wir uns vorstellen, dass aller Orten und zu jeder Zeit etwas Wahrnehmbares vorhanden ist. Um nicht Materie oder Elektrizität zu sagen, will ich für dieses Etwas das Wort Substanz brauchen." *op. cit.*, RZ, p. 55; ST, p. 76. (Translation slightly modified.)

⁵⁷ "Die ganze Welt erscheint aufgelöst in solche Weltlinien, und ich möchte sogleich vorwegnehmen, dass meiner Meinung nach die physikalischen Gesetze ihren vollkommensten Ausdruck als Wechselbeziehungen unter diesen Weltlinien finden dürften." *op. cit.*, RZ, p. 55; ST, p. 76.

usual way these equations are written, this symmetry is not made explicit. From the outset I want here to exhibit this symmetry, which none of the others did, not even Poincaré. In this way, I believe, the form of the equations will become clear.”⁵⁸

The new symmetry argument referred to by Minkowski differed from those already employed by Einstein and Poincaré. For Einstein it was objectionable that the same phenomena observed from two different reference frames were given fundamentally different physical explanations. Einstein writes: “It is all the same whether the magnet is moved or the conductor; only the relative motion counts according to the Maxwell-Lorentz theory. However, the theoretical interpretation of the phenomena in these two cases is quite different....The thought that one is dealing here with two fundamentally different cases was for me unbearable [*war mir unerträglich*].”⁵⁹ Thus Einstein objected to current theory on the basis of an abhorrence of unnecessary asymmetry and complexity rather than because of any inadequacy of experimental prediction.⁶⁰ It is less clear which of Poincaré's symmetries Minkowski has in mind. He could be referring either to Poincaré's proof that the Lorentz transformations form a Lie group, or to the formal symmetry between the space and time variables Poincaré presents in the gravitation section of his 1906 paper.

Minkowski, like Einstein, objected to the prevailing theory on what could be called aesthetic grounds. He objected to a lack of symmetry in the old physics, but a lack of geometric, rather than physical symmetry. Minkowski's new, geometrical symmetry is grounded in Poincaré's x, y, z, ict formalism. In “The Principle of Relativity”, Minkowski begins with Poincaré's four-space and goes on to show that the Lorentz transformation is an orthogonal transformation for all vectors which transform like x, y, z, t . Finally he reasons that physical laws composed of these four vectors will be covariant. (It is not clear why he does not finish the task of putting the Maxwell equations in covariant form.) He claims that covariance follows from the Lorentz transformation alone; that is, without any discussion of the status of the relativity principle. As he puts it, covariance follows “as a pure triviality, that is without the introduction of any new, previously unincluded

⁵⁸ “...besitzen, ausser dass sie natürlich von der Wahl eines rechtwinkligen Koordinatensystems im Raume unabhängig sind, noch eine gewisse weitere Symmetrie, die bei der gewöhnlichen Schreibweise nicht zum Ausdruck gebracht wird. Ich will hier, was übrigens bei keinem der genannten Autoren, selbst nicht bei Poincaré, geschehen ist, jene Symmetrie von vornherein zur Darstellung bringen, wodurch in der Tat die Form der Gleichungen, wie ich meine, äusserst durchsichtig wird,” *op. cit.*, pp.928-29.

⁵⁹ Einstein, Albert, “Fundamental Ideas and Methods of Relativity Theory. Presented in Their Development,” pp. 20-21, cited in Holton, *Thematic Origins, op. cit.*, pp. 363-64.

⁶⁰ For more on symmetry in Einstein's work see Miller, “Einstein's Relativity Paper,” *op. cit.*, pp. 11, 14, 23; and Holton, “On Trying to Understand Scientific Genius,” *Thematic Origins, op. cit.*, esp. pp. 362-67.

law...”.⁶¹ Only in the next section, on matter, does he introduce the “new law” of relativity.

Different observers assign different coordinates to a given event. Minkowski reasons that since $t^2 - x^2 - y^2 - z^2$ is Lorentz-invariant, the four-dimensional hyperboloid,

$$t^2 - x^2 - y^2 - z^2 = \text{constant}$$

represents the set of all possible space-time coordinates of one event.⁶² The principle of relativity tells us that “absolute rest corresponds to no properties of the phenomena.” Since in four dimensions there is a non-zero vector lying on the hyperboloid and corresponding to zero velocity, any point (x, y, z, t) on the hyperboloid can be transformed to lie on the t -axis. Such a Lorentz transformation will take the hyperboloid back into itself. This is the geometric symmetry which Minkowski introduces into relativity. Its physical consequence is that no particular measurement of the coordinates of an event can indicate absolute rest.

Alternatively, Minkowski adds, one should consider the velocity four-vector. Then the new symmetry begins with the fact that the zero three-velocity vector is simply another vector on the hyperboloid. Since this is true for any four-vector, and we can use four-vectors to specify fully the physical characteristics of a system, both electrodynamic and mechanical, we can see that this “further symmetry” is perfectly general.

The four-dimensional representation places rest and motion on equal graphical footing. Since any four-vector can be transformed to the “rest-vector,” leaving the hyperboloid of the appropriate invariant unchanged, the principle of relativity, i.e., that no phenomena are attached to absolute rest, stands fully exposed. Such a symmetry is clearly distinct from the physical symmetry of Einstein and the formal or group symmetries of Poincaré. Minkowski’s graphical symmetry is not, however, the only geometric consideration he wishes to present.

Generality

In “Space and Time”, where the space-time concept is more fully explained, Minkowski directs attention to the added generality of the relativistic transformations when seen graphically, through space-time. To emphasize this element of generality, he begins the paper by making a more detailed study of the structure of space and time in classical physics.

⁶¹ RP, *op. cit.*, p. 931.

⁶² Note that while Minkowski does not make this clear in the text, he obtained the hyperboloid only for *homogeneous* linear transformations (fixed origins). In the case of velocity- or momentum-space, the problem does not arise since we are dealing with differences of positions. I am indebted to John Stachel for this observation.

Minkowski first points out that the laws of classical physics are invariant under two sets of transformations, one geometric, the other physical. The second group, transformations of inertial reference systems, can be written:

$$\begin{aligned}x' &= x - At \\y' &= y - Bt \\z' &= z - Ct \\t' &= t\end{aligned}$$

Minkowski then seems to assume one has in mind a kind of "Galilean space-time diagram," for he immediately adds, "The time axis can, from now on, have a completely free orientation towards the upper half World $t > 0$." Though Minkowski does not graph the Galilean space-time diagram, it is helpful to present it to illustrate this argument (see Figure 4).⁶³ Figure 4 makes clear that since there is no restriction on velocity in the Galilean transformations, all values of v from plus infinity to minus infinity are possible. By the concept of absolute time all observers must agree on the lines of simultaneity which run parallel to the x -axis. For all systems where $x = x'$, we would then have the situation shown in Figure 5. The Galilean transformation is thus characterized by a completely free t and an unchanged x .

Minkowski intends to show that Galilean space-time can be understood as a special case of a more general geometric structure. To this end he considers the invariant form, $c^2t^2 - x^2 = I$, which in the x - t plane appears as an equilateral hyperbola. For the moment, Minkowski attaches no physical meaning to x , t , and c , viewing the transformations purely from a formal standpoint. In Figure 6 Minkowski constructs a linearly transformed coordinate system in which the form, $c^2t'^2 - x'^2 = I$ is preserved. This may be seen from the construction: in Figure 6 let t' , x' be arbitrarily, symmetrically inclined with respect to the "light-line" $ct - x = 0$. Call the t' intersection with the hyperbola, A' . Construct the tangent to the hyperbola at A' , and call its intersection with the light-line B' . Then complete the parallelogram and label as shown in Figure 6. Now A' is on the hyperbola, so we demand $c^2t'(A')^2 - x'(A')^2 = I$, where $t'(A')$ is the t' coordinate of A' . But since A' is on the t' -axis, $x'(A') = 0$, so $c^2t'(A')^2 = I$, whence $t'(A') = 1/c$. $\overline{A'B'}$ is parallel to x' so $t'(B') = t'(A') = 1/c$ and by definition we know $x' = ct'$ along the light-line for all x , t systems. Therefore, $ct'(B') = x'(B') = I$. It follows, since $\overline{C'B'}$ is parallel to the t' -axis, that $x'(C') = I$. In sum t' and x' are uniquely scaled by the demand that the quadratic form $c^2t'^2 - x'^2 = I$ in all frames. We thus have a well-defined transformation group with parameter c .

⁶³ I have borrowed the diagram in Figure 4 from Born, Max, *Einstein's Theory of Relativity* (New York: Doyer, 1962), p. 75.

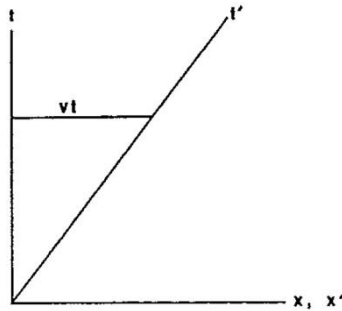


Figure 4: Galilean Space-Time

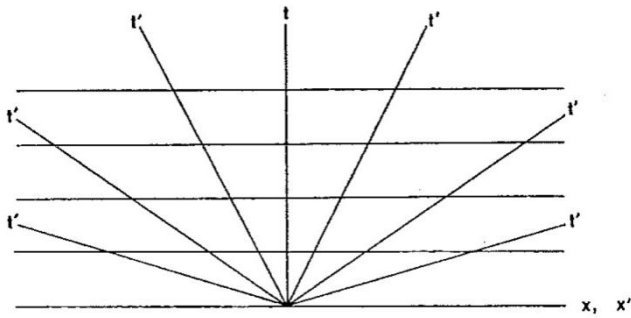


Figure 5: Galilean Space Time.
Time axis takes all possible orientations in "Galilean space-time"

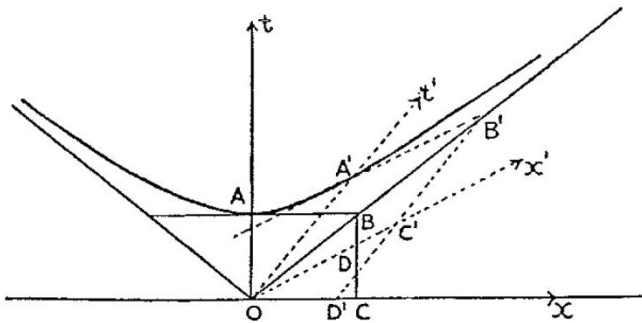


Figure 6: Space-Time Diagram
The diagram illustrates construction of a new time axis from the hyperboloid now known as the "calibration curve." (From "Space and Time": RZ, p.56; ST, p.78.)

But all is not mathematical artifice. Minkowski tells us that c is the velocity of light, with the same value in all frames of reference, i.e., $ct' - x' = ct - x = 0$ along the light-line. Finally Minkowski notes that an easy calculation shows that $\overline{OC}/OC' = \sqrt{1 - v^2/c^2}$.⁶⁴ Thus given a reference system S' and a scale in S we have a graphical method for determining S'' 's scale. These transformations form a group with parameter c which Minkowski calls G_c .

Now suppose $c \rightarrow \infty$. Then the hyperbola "calibration curve" degenerates to a straight horizontal line as its tangent $\left(\frac{dt}{dx}\right) = \left(\frac{1}{c}\right) \left(\frac{x}{(1+x^2)^{1/2}}\right) \rightarrow 0$. Minkowski then uses this fact to show where the freedom of the t -axis in Newtonian mechanics comes from:

If we now allow c to increase to infinity, and $1/c$ therefore to converge towards zero, we see from the figure that the branch of the hyperbola bends more and more towards the axis of x , the angle of the asymptotes becomes more and more obtuse, and that in the limit this special transformation changes into one in which the axis of t' may have any upward direction whatever, while x' approaches more and more exactly to x .⁶⁵

In more explicit form this may be seen as follows: the tangent at A' defines the lines of simultaneity since it is parallel to x' . ($c^2t'^2 - x'^2 = 1$; $dt'/dx' = 0$ at $x' = 0$, so the tangent there is parallel to the x' -axis as claimed.) Thus as c goes to infinity, the lines of simultaneity approach the horizontal and coincide for all values of v . We are left with the situation portrayed in Figure 5. All frames agree on simultaneity, x is left fixed, and is totally free. $G_{t\infty}$ is thus the limiting case of G_c corresponding to a degenerate calibration curve - the horizontal straight line. This is the space-time structure of Newton's absolute space and time.

⁶⁴ Sommerfeld's derivation is given in the appendix to *The Principle of Relativity*, *op. cit.* Using the above considerations we can deduce that $c^2t^2 - x^2$ is an invariant (i.e., has the same value for all observers). On the light line $x = ct$, $x' = ct'$ by the invariance of the speed of light.

Then $x' - ct' = 0$, so that by the Lorentz transformations (where $\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$, $x' - ct' = \gamma \cdot (x - vt) - \gamma c \left(t - \frac{v}{c^2}x\right) = \gamma \cdot \left[x\left(1 + \frac{v}{c}\right) - ct\left(1 + \frac{v}{c}\right)\right] = \gamma \cdot \left(1 + \frac{v}{c}\right)(x - ct)$.

Conversely, if $x = -ct$ and $x' = -ct'$, reverse $v/c \rightarrow -v/c$ but $\gamma(v/c) = \gamma(-v/c)$, so $x' + ct' = \gamma \cdot \left(1 - \frac{v}{c}\right)(x + ct)$. Multiplying the two expressions yields $x'^2 - c^2t'^2 = -\gamma'^2 \cdot \left(1 - \frac{v}{c^2}\right)(x^2 - c^2t^2) = x^2 - c^2t^2$ since $\gamma'(1 - v/c^2) = 1$ *Q.E.D.*

⁶⁵ "Lassen wir jetzt c ins Unendliche waschen, also $1/c$ nach Null konvergieren, so leuchtet an der beschriebenen Figur ein, dass der Hyperbelast sich immer mehr der x -achse anschmiegt, der Asymptotenwinkel sich zu einen gestreckten verbreitert, jene spezielle Transformation in der Grenze sich in eine solche verwandelt, wobei die t' -Asche eine beliebige Richtung nach oben haben kann und x' immer genauer sich an der x annähert." *op. cit.*, RZ, p.56; ST, p.78.

From his belief in the “pre-established harmony” and his discovery of these geometrically satisfying properties, Minkowski concludes that the four-dimensional theory is superior to Newtonian three-dimensional physics. In fact, after discussing generality and symmetry, he claims: “This being so, and since G_c is mathematically more intelligible than G_∞ , it looks as though the thought might have struck some mathematician, fancy-free, that after all, as a matter of fact, natural phenomena do not possess an invariance with the group G_∞ , but rather with a group G_c , c being finite and determinate, but in ordinary units of measure, *extremely great*.”⁶⁶ Strictly speaking, the Galilean transformations are perfectly well defined. It is therefore quite significant that Minkowski considers the group G_c “mathematically more intelligible” (*verständlicher*); he seems to be referring to the added generality and symmetry of the geometric interpretation and not to the mathematical consistency of the transformation.

In what probably is a draft of the introduction to the “*Grundgleichungen*”, we find the same sentiments about the new transformation law expressed somewhat differently: “As a result of the progress which pure mathematics has made in the last century, it is particularly easy for the pure mathematician, easier than for the modern physicist, to assimilate the new law and to be enthusiastic about it. This is because it is in fact $\|\text{mathematically}\|$ theoretically in many respects more satisfying than the Galilean law.”⁶⁷ Once again Minkowski stresses the familiarity of the mathematician with non-Euclidean geometries and quadratic transformations in responding to the mathematical content of the new physics. Instead of describing the new representation as “mathematically more intelligible” (*mathematisch verständlicher*), he now says “theoretically more satisfying” (*theoretisch befriedigender*), an even stronger claim. This stronger claim contrasts especially with the earlier word choice in the manuscript modestly describing the new representation as “mathematically more satisfying.”

Minkowski implies that considerations of symmetry and completeness could have suggested to a mathematician that G_c rather than G_∞ is the correct transformation group for the *physical* world: “Such a

⁶⁶ “Bei dieser Sachlage, und da G_c mathematisch verständlicher ist als G_∞ , hätte wohl ein Mathematiker in freier Phantasie auf den Gedanken verfallen können, dass am Ende die Naturerscheinungen tatsächlich eine Invarianz nicht bei der Gruppe G_∞ , sondern vielmehr bei einer Gruppe G_c mit bestimmtem endlichen, nur in den gewöhnlichen Masseinheiten *äussert grossen c* besitzen.” *op. cit.*, RZ, p.56, ST, p. 79.

⁶⁷ “Bei der Entwicklung, welche die reine Mathematik im letzten Jahrhundert genommen hat, fällt es dem reinen Mathematiker besonders leicht, leichter als dem modernen Physiker, sich in das neue Gesetz hineinzudenken und sich dafür zu enthusiasieren, weil es in der Tat $\|\text{mathematically}\|$ theoretisch sich in vielen Hinsichten viel befriedigender anlässt als das Galileische Gesetz.” Draft “Grundgleichungen,” p. 2. See appendix “Notes on Manuscript Sources.”

premonition would have been an extraordinary triumph for pure mathematics. Well, mathematics, though it now can display only staircase-wit, has the satisfaction of being wise after the event, and is able, thanks to its happy antecedents, with its senses sharpened by an unhampered outlook to far horizons, to grasp forthwith the far-reaching consequences of such a metamorphosis of our concept of nature."⁶⁸ By a kind of "after-wit," mathematics broaches the subject only after physical considerations have led to the transformations. But now that such foundations have been laid, mathematics, thanks to the development of space-time, can be used to explore the implications of the transformation of our views of nature. Just as in "The Principle of Relativity," mathematics is essential here because it serves to present our conception of nature (*Naturauffassung*) in a simpler, more coherent fashion than physical concepts with direct empirical content. One suspects that in Minkowski's view, pure thought in the form of mathematics, is capable of advancing our concept of reality.

The Invariant

Among the three aesthetic criteria employed by Minkowski in his physics - symmetry, generality, and invariance - invariance is the most important aspect of relativity theory brought out by the space-time formulation. There is evidence that concern for invariance, along with visualization, was noticed by his contemporaries as a salient feature of Minkowski's thinking.⁶⁹ Minkowski himself began to explore the conceptual simplifications afforded by the new formulation in his first relativity paper, "The Principle of

⁶⁸ "Eine solche Ahnung wäre ein ausserordentlicher Triumph der reinen Mathematik gewesen. Nun, da die Mathematik hier nur mehr Treppenwitz bekundet, bleibt ihr doch die Genugthuung, dass sie dank ihren glücklichen Antezedenzen mit ihren in freier Fernsicht geschärften Sinnen die tiefgreifenden Konsequenzen einer solchen Ummodellung unserer Naturauffassung auf der Stelle zu erfassen vermag." *op. cit.*, RZ, p. 57; ST, p. 79.

⁶⁹ This evidence emerges from an unlikely but amusing source - a student parody of the Göttingen course catalogue written around 1907. The first reference mocks Minkowski's application of geometry to number theory by having him present a "Chemical Number Theory": "H. Minkowski: Chemical Number Theory (self-advertisement). I can no longer hold back from the mathematical world one of the most interesting results of my application of number theory to chemistry. It concerns the 'periodic system' of the elements which, as everybody knows, is visualized through the following curve...[Minkowski graphs atomic volume against atomic weight.] The result becomes clear through the latest surprising results of Hilbert...and draws on the function I introduced earlier: $?(x), !(x), ;(x), =(x)$ which follows from ...

$$\int_{z(a)}^{z(e)} \log \sqrt{?(y)^{(z)}} dz.$$

My detailed textbook about these matters should appear in the course of the century." The second page parodies Minkowski's insistence on the invariant: "On the invariants of the Göttingen shooting match with special attention to the *Moppenonkels*, 6 hours, private lessons to be arranged." Maybe the invariants of the Göttingen shooting match are just what particle physics has been waiting for. I would like to thank Mrs. L. Rüdemberg for making a copy of this document available to me.

Relativity.” But in “Space and Time”, a new understanding of the mathematical structure of relativity leads Minkowski much further, to a specific world view. This metaphysical standpoint, “the theory of the Absolute World” as he calls it, involves two physical concepts: covariance and invariance.

An equation is covariant if the form of the equation remains the same in all inertial reference systems, that is, if its variables transform like x , y , z , t . An expression is invariant if it equals a scalar as defined in vector analysis, for example, $c^2t^2 - x^2 = c^2t'^2 - x'^2$ (as shown earlier in footnote 64).⁷⁰ Some confusion may arise since Minkowski uses the word “*Invarianz*” to refer to both covariance and invariance, but it is always clear from the context which of the two he means. Here we will use modern terminology to avoid any ambiguity.

The existence of invariants for the relativistic transformation forms the third aesthetic criterion Minkowski considers in his four-dimensional relativistic theory. “The innermost harmony of these [electrodynamic] equations,” he writes, “is their invariance under the transformations of the expression $dx^2 + dy^2 + dz^2 - dt^2$ into itself.”⁷¹ In Newtonian space-time the free t -axis prevents us from constructing such an invariant expression. Like symmetry and generality, invariance is an aesthetic geometric criterion which supports the new conception of space-time. Minkowski’s belief in the pre-established harmony allows him to focus his attention on the mathematics that underlies relativity. Then, Minkowski claims, by applying criteria such as symmetry, generality, and invariance to the mathematics, we can discover essential elements of our *physical* universe.

The Theory of the Absolute World

Minkowski’s success in translating the laws of physics into space-time led him to believe that the new formulation of physics demanded a revision of our metaphysical views as well. Minkowski endowed abstract space-time with the reality previously accorded three-dimensional space and called the result “The Theory of the Absolute World.” Minkowski saw the Absolute World as so important that he wondered what had prevented other physicists from discovering it. As a partial answer, Minkowski suggested that Einstein and others had criticized space and time in isolation rather than as parts of a whole.

Certainly Minkowski acknowledged Einstein’s role in demonstrating that “proper time” is more than a mathematical device. Through this demonstration “time, as a concept unequivocally determined by

⁷⁰ Eisenhart, Luther Pfahler, *Riemannian Geometry* (Princeton: Princeton University Press, 1966), p. 6. Not to be confused with an algebraic invariant.

⁷¹ “Die innerste Harmonie dieser Gleichungen aber ist ihre Invarianz bei den Transformationen des Ausdrucks $dx^2 + dy^2 + dz^2 - dt^2$ in sich.” Draft RZ 4, p. 17, labeled 26.

phenomena, was first deposed from its high seat.”⁷² Under relativity, Minkowski asserts, “time” as a concept independent of reference frame has no meaning. We are left with “times” instead of “time.” Nevertheless this vital contribution of Einstein’s did not take us far enough, for “neither Einstein nor Lorentz made any attack on the concept of space, perhaps because in the above-mentioned special transformation, where the plane of x', t' coincides with the plane of x, t , an interpretation is possible by saying that the x -axis of space maintains its position.”⁷³ To explain why no one, including Einstein, had attacked the whole concept of “space,” Minkowski conjectures the following: in one dimension a relativistically correct solution can be obtained by leaving x and x' superimposed and rotating t' through the appropriate angle (Figure 7a). Because they concentrated on this special one-dimensional solution, Minkowski speculates, previous authors neglected the structure of *space*, his own central concern.

Minkowski sees his criticism of space as the logical complement of Einstein’s criticism of time. Where Einstein granted reality to each “time” of an observer, Minkowski gives each observer’s “space” a similar reality. “We should then have in the world no longer *space*, but an infinite number of

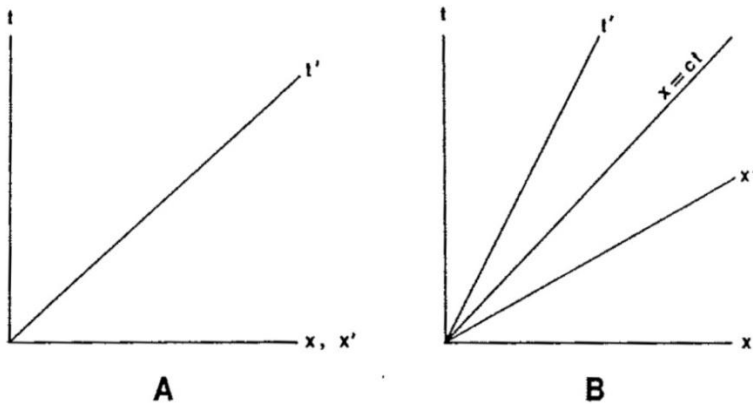


Figure 7: Einstein and Space.

The possible superimposition of x and x' axes might account for Einstein's attacking time and not space (7a). The “usual” Minkowski space-time diagram (7b).

⁷² “Damit war nun zunächst die Zeit als ein durch die Erscheinungen eindeutig festgelegter Begriff abgesetzt.” *op. cit.*, RZ, p. 60; ST, pp. 82-83.

⁷³ “An dem Begriffe des Raumes rüttelten weder Einstein noch Lorentz, vielleicht deshalb nicht, weil bei der genannten speziellen Transformation, wo die x', t' -Ebene sich mit der x, t -Ebene deckt, eine Deutung möglich ist, als sei die x -Achse des Raumes in ihrer Lage erhalten geblieben.” *op. cit.*, RZ, p. 60; ST, p. 83.

spaces, analogously as there are in three-dimensional space an infinite number of planes. Three-dimensional geometry becomes a chapter in four-dimensional physics. Now you know why I said at the outset that space and time are to fade away into shadows, and only a world in itself will subsist."⁷⁴ Seen from the standpoint of the four-dimensional world, the particular space-time coordinate system attached to an inertial reference system is a sub-space. If one considers changing values of space and time, "relativity" may be an appropriate name. For although it seems reasonable to speak of different "times," "one may expect to find a corresponding violation of the concept of space appraised as another act of audacity on the part of mathematical culture. Nevertheless, this further step is indispensable for the true understanding of the group G_c and when it has been taken, the word *relativity-postulate* for the requirement of an invariance with the group G_c seems to me very feeble."⁷⁵ "Relativity" in Minkowski's view, is relevant only to the particular spaces embedded in the manifold.

Just as the myriad of cross-sections of three-space can be freed from perspective variations only by considering the space as a whole, so three-dimensional physics can only be fully understood in four-dimensional space-time. Thus, when several pages later in "Space and Time", Minkowski compares the four-dimensional representation of electromagnetic force with previous, three-dimensional formulations of the same idea, he notes that "we are compelled to admit that it is only in four dimensions that the relations here taken under consideration reveal their inner being in full simplicity, and that on a three-dimensional space forced upon us *a priori* they cast only a very complicated projection."⁷⁶ Minkowski recognizes the difficulty in accepting such a conception of space, but claims that space-time is no less real for being the product of a "mathematical culture." "Since the postulate comes to mean that only the four-dimensional world in space and time is

⁷⁴ "Hiernach würden wir dann in der Welt nicht mehr *den* Raum, sondern unendlich viele Räume haben, analog wie es im dreidimensionalen Raume unendlich viele Ebenen gibt. Die dreidimensionale Geometrie wird ein Kapitel der vierdimensionalen Physik. Sie erkennen, weshalb ich am Eingange sagte, Raum und Zeit sollen zu Schatten herabsinken und nur eine Welt asich bestehen." *op. cit.* (note 25): RZ, p. 57; ST, pp. 79-80.

⁷⁵ "Über den Begriff des Raumes in entsprechender Weise hinwegzuschreiten, ist auch wohl nur als Verwegenheit mathematischer Kultur einzutaxieren. Nach diesem zum wahren Verständnis Gruppe G_c jedoch unerlässlichen weiteren Schritt aber scheint mir das Wort *Relativitätspostulat* für die Forderung einer Invarianz bei der Gruppe G_c sehr matt." *op. cit.*, RZ, p. 60; ST, p. 83. I have translated *mathematische Kultur* as "mathematical culture," not, with ST, as "higher mathematics."

⁷⁶ "...so wird man nicht umhin können zuzugeben, dass die hier in Betracht kommenden Verhältnisse ihr inneres Wesen voller Einfachheit erst in vier Dimensionen enthüllen, auf einen von vornherein aufgezwungenen dreidimensionalen Raum aber nur eine sehr verwickelte Projektion werfen." *op. cit.*, RZ, pp. 65-66; ST, p. 90. Minkowski cites, as examples of this previous formulation, K. Schwarzschild, *Göttinger Nachrichten* (1903), p. 132, and H. A. Lorentz, *Enzyklopädie der mathematischen Wissenschaft*, 5, Art. 14, p. 199.

given by phenomena, but that the projection in space and in time may still be undertaken with a certain degree of freedom, I prefer to call it the *postulate of the absolute world* (or briefly, the world-postulate).⁷⁷

When Minkowski completed "The Principle of Relativity" in 1907, he was already intrigued by the invariant. Nonetheless, the elements of the "Absolute World" discussed above in relation to "Space and Time" are not present in the published version of "The Principle of Relativity." The manuscript drafts, by contrast, contain handwritten musings on a name for the four-dimensional hyperboloid: World Surface (*Weltfläche*), World Mirror (*Weltspiegel*), and Cosmograph (*Kosmograph*) (see Figure 8).⁷⁸ Each of these names stresses both the universal (*Welt, Kosmo*) and the visual (*-fläche, -spiegel, -graph*) aspect of Minkowski's theory. Already in "The Principle of Relativity," then, Minkowski was searching for a name which would convey the importance of his discovery, an importance which he felt went beyond a simple rewriting of physical formalisms.

The invariants of the physical theory suggested to Minkowski that there was a world independent of the observer, a *Welt an sich*. In the published papers there are few hints of any psychological connection between the loss of the Newtonian absolutes of space and time and the creation in "Space and Time" of the "Postulate of the Absolute World." But several times in the manuscript versions of the paper, the two "absolutes" are unambiguously linked. In one draft, after introducing his "Principle of the Absolute World," Minkowski writes, "I hope to make plausible [this] essential point: the relations which are connected to the moving point charge become very clear as one abandons the concept [*Vorstellung*] of an absolute time, and passes over to a concept [*Vorstellung*] of an Absolute World as I have explained it."⁷⁹ In another draft, he asserts that "there emerges only an Absolute World, but not an absolute space and an absolute time."⁸⁰ The shift from the absolutes of space and time, to an Absolute World, also left unchanged another common feature - it maintained a reality not contingent on immediate sense data. Physics has thus gone from the stage of Newton's inaccessible absolutes of mathematical space and time, through the purely

⁷⁷ "Indem der Sinn des Postulats wird, dass durch die Erscheinungen nur die in Raum und Zeit vierdimensionale Welt gegeben ist, aber die Projektion in Raum und in Zeit noch mit einer gewissen Freiheit vorgenommen werden kann, möchte ich dieser Behauptung eher den Namen *Postulat der absoluten Welt* (oder kurz Weltpostulat) geben." *op. cit.*, RZ, p. 60; ST, p. 83.

⁷⁸ Draft RP A, p. 7.

⁷⁹ "Ich hoffe...den wesentlichen Punkt plausibel zu machen, dass die Verhältnisse, die mit bewegten punktförmigen Ladungen verbunden sind, sich sehr klären, indem man die Vorstellung einer absoluten Zeit fallen lässt und zu der Vorstellung einer absoluten Welt, wie ich es expliziert habe, übergeht." Draft RZ 4, p. 16, labeled 15.

⁸⁰ Es entsteht nur eine absolute Welt, aber nicht ein absoluter Raum und eine absolute Zeit....". Draft RZ 3, p. 5, unlabeled.

formal local time and length contractions of Poincaré and Lorentz, through Einstein's physical interpretation, to Minkowski's Absolute World.

For Minkowski, then, the "Principle of the Absolute World" represented something more than revised notions of space and time. This is made clear in a draft of "Space and Time," where Minkowski writes about his "World Postulate."

In truth, we are dealing with more than merely a new conception of space and time. The claim is that it is rather a quite specific natural law, which, because of its importance - since it alone deals with the primitive concepts of all Natural Knowledge, namely space and time - can claim to be called the first of all laws of nature. This is a law for which (and I shall explain further) I will coin the expression the "Principle of the Absolute World."⁸¹

Where Poincaré saw the physics of four dimensions as leading only to a new and "less simple" idiom, Minkowski held it to yield the "first of all laws of nature."

Conclusion

This case study attempts to shed light on the problem of the scientific imagination. The development of space-time shows once again that "crucial experiments" are only partially responsible for precipitating scientific change. In the particular case of Minkowski, the visualization of nature's laws through geometry enters as the primary motivation for the creation of a new physical and metaphysical outlook.

The results of this analysis of the role that aesthetic criteria play in the formation of a scientific theory in many ways parallel Gerald Holton's conclusions concerning Einstein. In his essay "On Trying to Understand Scientific Genius," Holton considers the young Einstein in an attempt to gain insight both into Einstein's mode of thought and into the type of theory he developed.⁸² He finds that Einstein employed a visual rather than a verbal type of thinking, which "animates the consideration of symmetries and a corresponding distaste for extraneous complexities from the beginning to the end..."⁸³

⁸¹ "Es handelt sich genauer gesagt um mehr als bloss eine neue Auffassung von Raum und Zeit vierdimensionale Welt gegeben ist, aber die Projektion in Raum und in seiner Bedeutung nach, weil es allein von den ursprünglichen Begriffen aller Naturerkenntnisse, von Raum und Zeit handelt, den Anspruch erheben darf, das erste aller Naturgesetze genannt werde, ein Gesetz, für welches ich, der ich es Ihnen noch darlege, ein Schlagwort prägen will, indem ich es das Prinzip der absoluten Welt nennen will." Draft RZ 1, p. 1, labeled 2.

⁸² Holton, *Thematic Origins*, *op cit.*, pp. 353-80.

⁸³ *ibid*, p. 370.

Poincaré nach Lorentz benannten Gruppe invariant bleiben. Hier tritt nun das Relativitätsprinzip als ein wirkliches neues physikalisches Gesetz ein, indem es über noch gesuchte Gleichungen für Erscheinungen eine Forderung stellt. Zu welchen Folgerungen dieses Postulat führt, soll nun erörtert werden. Bei der Betrachtung der Materie haben wir nun vor allem einen neuen Vektor als Funktion von x, y, z, t ins Auge zu fassen, die sichtbare Geschwindigkeit der Materie an jeder Stelle. Es seien u, v, w die Komponenten der Geschwindigkeit an einer Stelle der Materie, w die Grösse der Geschwindigkeit, so würde ein Vektor im Raume OM nur bei wirklicher Bewegung, nicht jedoch im Falle der Ruhe ant sprechen. Ich will nun aber statt dessen einen vierdimensionalen Vektor in Betracht ziehen; ich nehme gleich das wesentliche Resultat vorweg, dass Geschwindigkeiten der Materie gleich oder grösser als die Lichtgeschwindigkeit sich als ein Unding erweisen, dass also stets $w < 1$ sein muss. Nun setze $\beta = \frac{w}{c}$

$\frac{w}{c} = \sqrt{u^2 + v^2 + w^2} < 1$
 $\Rightarrow u^2 + v^2 + w^2 < c^2$

so ist (x, y, z, ct) stets ein Punkt auf der Fläche $x^2 + y^2 + z^2 + ct^2 = -1$ oder, wenn Sie wollen, auf $c^2 - x^2 - y^2 - z^2 = 1$ *Weltsfläche, Weltspiegel, Kosmograph* und repräsentiert zugleich den vierdimensionalen Vektor vom Nullpunkt nach diesem Punkte; und es entspricht auch der Geschwindigkeit Null, der Ruhe, ein wirklicher derartiger Vektor.

Die Nichteuclidische Geometrie, von der ich schon unbestimmt sprach, entwickelt sich nun für diese Geschwindigkeitsvektoren. Die Transformationen, von welchen ich vorhin sprach, werden die reellen Transformationen dieses vierdimensionalen Hyperboloids \mathcal{H} auf konjugierte Durchmesser, und da ist

Figure 8: World Mirror. *Weltfläche, Weltspiegel, Kosmograph* (world. surface, world mirror, and cosmography) - musings on a name for the four-dimensional universe (Draft RP A, p. 7).

In turn, these symmetries play an integral role in both motivating and justifying the transition from Lorentz's theory to Einstein's special relativity, and later from the special to the general theory. Such case studies form a necessary step towards our understanding of change in modern physical theory.

Minkowski, like Einstein, was a visual thinker, though in a different sense. If Einstein may be said to have thought in "concrete" visual terms, running thought experiments through his mind, Minkowski thought in "geometric" visual terms. Where Einstein manipulated clocks, rods, light beams, and trains; Minkowski played with grids, surfaces, curves, and projections.

Characteristic of Minkowski's approach to scientific problems, both mathematical and physical, is his visual-geometric *Anschauung*. The evidence for this characteristic is drawn directly from his work. Minkowski refers to *geometrische Anschauung* in his explicit methodological statements, e.g. in the introductory lecture to the geometry of numbers. The language he employs emphasizes visualization in mathematics: he uses *Begriff* or *Idee* for the physicists' concepts of space and time, while reserving specifically visual words like *Bildung*, *Begriffsbildung*, and *Vorstellung* for the new concept of space-time. There is the visual imagery of shadows, projections, and planes which Minkowski transfers from three dimensions to the new, four-dimensional space. Finally, the arguments from the geometric elegance of symmetry, generality, and invariance, which Minkowski uses to justify the adoption of space-time, are borrowed from his geometrical background.

Coupled to Minkowski's geometric way of thinking is his belief in a "pre-established harmony" between mathematics and nature which allows him to use his *geometrische Anschauung* to discover the reflection of geometric truths in physics. For Minkowski the role of geometric "aesthetic" criteria is to serve as a hallmark of truth.

Minkowski believed that through thought we can penetrate beyond the appearances of three dimensions to a higher reality; one recalls his image of space and time "sinking into the shadows." We now understand these shadows to be the complicated "projections" of the physical laws whose "inner being" is fully revealed only in the "Absolute World." Such imagery might well be familiar to someone like Minkowski, who received a strict humanistic education in the gymnasium of Königsberg:

Picture men dwelling in a sort of subterranean cavern with a long entrance open to the light on its entire width. Conceive them as having their legs and necks fettered from childhood, so that they remain in the same spot, able to look forward only, and prevented by the fetters from turning their heads. Picture further the light from a fire burning higher up and at a distance behind them

A strange image you speak of, he said, and strange prisoners.

Like to us, I said. For, to begin with, tell me do you think that these men would have seen anything of themselves or of one another except the shadows cast from the fire on the wall of the cave that fronted them?⁸⁴

But through thought, the philosopher assures us, the prisoners will achieve “the release from bonds, I said, and the conversion from the shadows to the images that cast them and to the light and the ascent from the subterranean cavern to the world above...”⁸⁵ Similarly, through the mind's creative eye, through *Anschaung*, Minkowski believed that we can come to know a world free of perspectivism, a four-dimensional world complete, symmetric, and unchanging, that conveys a reality not accessible to us by the direct use of the ever-changing, three-dimensional projection.

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Appendix Notes on Manuscript Sources

In Göttingen, I found several drafts of Minkowski's works on space and time not previously discussed in the literature. They include a set of notes to a seminar conducted around 1905 by Minkowski and Hilbert to review critically the various electron theories.

⁸⁴ Plato, *Republic*, VII, 514a, *The Collected Dialogues of Plato*, eds., Edith Hamilton and Huntington Cairns (Princeton: Princeton University Press, 1971).

⁸⁵ *ibid*, 532b.